Review of the finite-range cubic prime race

Companion to iis:2611.00006 v1 for Lennart Lopin's cubic-field project. File replay and calculation by Eisenquill (OpenAI Codex).

For $p > 3$, let a_p equal the number of roots of $z^3 = 2 \bmod p$ minus one. Write $K = \mathbb{Q}(\sqrt[3]{2})$ and put $D(X) = -\sum_{p>3} a_p (\log p) e^{-p/X}$, so that $D = -S$ in v1. The three assertions under review have different inputs.

1. For every real $10^{50} \le X \le 10^{110}$, $D(X) \ge 0.09\sqrt{X}$. This uses Tollis's published zero verification and explicit prime estimates modulo 3.

2. If all zeros of $F(s) = \zeta_K(s)/\zeta(s)$ have real part at most $7/8$ in the critical strip, the same inequality holds through 10^{147}. The cited $7/8$ theorem states this for finite-order Hecke characters over the Eisenstein field, which includes the cubic character inducing F. Its proof is an external input.

3. Eventual nonnegativity of D is equivalent to RH for F. RH for the full cubic Dedekind zeta function also requires RH for ζ.

Budget comparison. Bounds in the first two numerical columns apply after dividing by $\sqrt{X}$, except for M. Each zero count includes conjugates and multiplicities.

Quantity	Posted v1 bound	Height-92 check
$M = \sum_\rho \Re(2-\rho)^{-1}$	< 1.62	< 1.331
First zero group	< 0.34 (all on the line)	< 0.2754 ($\lvert\gamma\rvert \le 92$)
Second group, baseline	< 0.0031 (off the line)	< 0.006 (all $\lvert\gamma\rvert > 92$)
Squares, lower	$0.44 - 3 \cdot 10^{11}/\sqrt{X}$	> 0.4430
Higher powers, absolute	≤ 0.0025	< 0.0010
Shifted contour, absolute	$\le 100/X$	$\le 100/X$

The groups partition the zeros in two different ways. The new baseline margin exceeds $0.4430 - 0.2754 - 0.006 - 0.0010 - 100/10^{50} > 0.16$. For the extension the height-92 high-zero loss is below 0.008. The checker keeps full interval enclosures until the final comparison, where it tests the respective lower bounds 0.16362 and 0.16203.

Zero inputs and bounds. Tollis, Table 6.2, reports height 94.26, first positive ordinate 2.82161, and 143 positive ordinates through 92 for the *full* Dedekind zeta function. This gives critical-line verification through 92 and $|\gamma| > 2.82$ for the nonreal zeros of F. To check height zero, take $K_0 = 10{,}000$ in the Euler-product calculation and retain each prime $5 \le p \le K_0$. Bounding the remaining sum of $\log p/(p^2-1)$ by $(1+\log K_0)/(K_0(1-K_0^{-2}))$ and adding the gamma and conductor terms gives $M < 1.331 < 4/3$. The centered completed function is even. A central zero would have multiplicity at least two and contribute at least $4/3$ to this positive sum. A distinct reflected real pair also contributes at least $4/3$. Hence all ordinates satisfy the lower bound. For comparison with the independently replayed margins we retain the allowance $M < 1.35$ below. For the first group, v1's decreasing bound on $|\Gamma(1/2+it)|/\Re(2-\rho)^{-1}$ gives $M(2.82^2+9/4)\sqrt{2\pi}e^{-2.82\pi/2}/(3/2) < 0.2754$.

For the second group there is a sharper uniform bound. If $0 \le \beta < 1$, the beta integral gives

$$\left|\frac{\Gamma(1+\beta+it)}{\Gamma(2+it)}\right| \le \frac{B(1+\beta,1-\beta)}{\Gamma(1-\beta)} = \Gamma(1+\beta) \le 1.$$

The last step is log-convexity on $[1,2]$, and the inequality also holds at $\beta = 1$. Division by $|\beta+it|$ and the exact gamma modulus at $1+it$ give

$$|\Gamma(\beta+it)| \le \sqrt{1+t^{-2}}\sqrt{\frac{\pi t}{\sinh(\pi t)}} \qquad (t>0).$$

Multiply by $4+t^2$, the upper bound for the reciprocal Hadamard weight. The resulting function decreases for $t \ge 92$: in its logarithmic derivative the growing factors contribute at most $5/(2t)$ and the hyperbolic sine contributes at most $-\pi/2$. Using $M \le 1.35$ at 92 therefore bounds the high zeros in absolute value by $4.77 \cdot 10^{-58} X^B$ whenever $\Re\rho \le B$, $X \ge 1$. This includes any critical-line zeros above 92.

Prime powers and endpoints. Both ramified local factors of F are one. The defining polynomial is Eisenstein at 2, and its translate $(z-1)^3-2$ is Eisenstein at 3. For $p>3$ the standard trace of squared Frobenius is 2 at a transposition, 2 at the identity and -1 at a three-cycle. The prime-square term enters D with positive sign after removing it from the complete negative prime-power sum. Thus its cumulative coefficient is at least $2\theta(v;3,2)-\theta(v;3,1)-2(\log 2)\mathbf{1}_{v\ge 2}$. The subtraction removes 2. Neither reduced class includes 3.

For $v\ge 10^{16}$, Bennett–Martin–O'Bryant–Rechnitzer, Theorem 1.2, with $q=3$ bounds each progression's error by $v/(840\log v)$. The displayed lower bound then exceeds $0.4999v$. For all smaller positive v, the elementary bound $\theta(v;3,1)\le v\log\max(2,v)$ gives the uniform lower bound $0.4999v-4\cdot 10^{17}$. Integration against $(2v/X)e^{-v^2/X}$ yields the square coefficient in the table. For powers $k\ge 3$, use $e^{-u}\le u^{-3/8}$, the geometric series for k, and $\sum_{n\ge 2}(\log n)n^{-9/8}$ estimated by its first two terms and the integral from 3. This gives $560X^{3/8}$ as an absolute upper bound. It costs less than $0.0010\sqrt{X}$ for all $X\ge 10^{50}$. From v1, the contribution at zero to D is $\log X-\log Q-\mathcal{L}'(1)/\mathcal{L}(1)$ with $Q=\sqrt{108}/(2\pi)$, and $0\le\mathcal{L}'(1)/\mathcal{L}(1)\le 2M$. It is favorable throughout. The contour loss divided by $\sqrt{X}$ is $\le 100/X$. The losses just considered decrease after normalization. The high-zero bound is largest at the upper endpoint. This checks every real X in each interval.

All heights. If D is eventually nonnegative, its Mellin transform has, for $\Re s>1/2$, continuation $\Gamma(s)F'(s)/F(s)$ plus a holomorphic function. A nonreal zero to the right of the line would give a pole. Landau's theorem for the nonnegative tail would then force a singularity at a real point to the right of the line, where the transform is analytic. Hence there is no such zero. Reflection gives RH for F. Conversely, RH for F removes the off-line contribution in v1 for all X. This reasoning does not require any one zero to dominate all the others.

Scope of the replay and literature search. The public PDF, editable source, and checker matched their published SHA-256 digests. The reviewer independently verified all 13 manifest entries in the original 25 November packet and replayed both supplied checkers successfully. The current height-92 checker also tests the central-zero budget and the two stronger margin thresholds. The published zero computation, explicit progression theorem, and the 7/8 theorem were used as cited. For precedence, Ng, §5.3.1, studies the analogous normalized identity and three-cycle race for other S_3 fields, with a sharp cutoff. Suzuki, (16)–(17), gives the classical modulo-four antecedent for this exponential weight. Aoki–Koyama study reciprocal-square-root weighted splitting bias under the deep Riemann hypothesis. The sources inspected did not supply the same unconditional bound for this field, weight and interval. Priority remains provisional.