A squarefree denominator for the cubic prime count

Research checkpoint for $\mathbb{Q}(\sqrt[3]{2})$, 24 November 2026

Write $\mathcal{B}$ for the primes $q > 3$ where $t^3 = 2 \bmod q$ has no solution, and $R_N = \prod_{q \le N,\, q \in \mathcal{B}} q$. The previous checkpoint realizes R_N as the radical of a large common denominator of elliptic jets. Here we realize it as an ordinary least common denominator and test an analytic estimate for its coefficients. This uses a different series computed from the three ideal lattices. The square-root estimate in section 2 is conditional only on Dirichlet GRH. It applies to a signed sum over integers. The error term for the cubic primes is unchanged.

1. Constructing the coefficients from ideal counts

In $k = \mathbb{Q}(\sqrt{-3})$, let $a_j(n)$ count integral ideals of norm n prime to 6 in ray class j modulo 6. These can be computed by enumerating $u^2 - uv + v^2 = n$ at the coordinate pairs (u, v) congruent to $(1,0), (4,3), (1,3)$ modulo 6, respectively. The unique generator congruent to 1 modulo 3 fixes the multiplicity. Conjugation gives $a_1 = a_2$. Put $A = a_0 + a_1 + a_2$ and $a = a_0 - a_1$; $A(1) = 1$. Solve the integer Dirichlet convolution $A * b = a$ in increasing order, and put $f(n) = \mu^2(n) b(n)$. No prime splitting test is needed for this construction.

The Dirichlet series for b is $F(s) = L^{(6)}(s, \rho)/\zeta_k^{(6)}(s)$, with the Euler factors at 2 and 3 omitted. Here ρ is the standard S_3 representation, or equivalently the corresponding Hecke L-function over k. For $\Re s > 1$ its local factors, with $v = q^{-s}$, are

$$F_q(s) = \begin{cases} 1 & q \text{ has one or three cube roots of 2,} \\ (1-v)^3/(1-v^3) & q \in \mathcal{B}. \end{cases}$$

Indeed the two linear Euler factors over a split prime of k have character values $(1,1)$ or (ω, ω^2); an inert prime gives the same norm-q^2 factor on both sides. Thus $f(1) = 1$, $f(n) = (-3)^{\omega(n)}$ for squarefree n supported on $\mathcal{B}$ (here $\omega(n)$ counts distinct primes), and $f(n) = 0$ otherwise. For every positive integer N,

$$\operatorname{lcm}_{1 \le n \le N} \operatorname{den}\left(\frac{f(n)}{n}\right) = R_N.$$

Each nonzero denominator is squarefree with the required support, and the term $f(q)/q = -3/q$ forces each prime into the least common multiple at degree q. In particular the large repeated powers in the elliptic series have been removed without taking a radical after computing its coefficients. This identity alone gives no upper bound on R_N.

Numerical check. The accompanying `check_squarefree_series.py` enumerates lattice points, performs the integer convolution division and checks every least common denominator through 10,000 against a separate exhaustive cube-root test. There are 411 denominator primes. It also checks the last Euler-product identity in section 3 coefficient by coefficient. All these computations use the Python standard library.

2. A signed square-root estimate and a sieve test

For the analytic arguments omit 7 also, writing $f_*(n) = 0$ if $7 \mid n$ and $f_*(n) = f(n)$ otherwise. This changes R_N by just its known factor 7 when $N \geq 7$. It removes an artificial zero of the squarefree Euler factor $1-3\cdot 7^{-s}$ in the half-plane of interest. Put $F_*(s) = L^{(42)}(s,\rho)/\zeta_k^{(42)}(s)$. For $\Re s > 1$ the Dirichlet series for f_* equals

$$H(s) = F_*(s) \prod_{\substack{q\in\mathcal{B}\\ q\neq 7}} \frac{(1-3q^{-s})(1-q^{-3s})}{(1-q^{-s})^3}.$$

The correction factors are $1+O(q^{-2\sigma})$, $\sigma = \Re s$, and are nonzero on $\sigma > 1/2$, since the remaining q are at least 13. The product and its inverse are bounded on each closed half-plane $\sigma \geq 1/2+\delta$. Under Dirichlet GRH $\zeta_k = \zeta L(\chi_{-3})$ is zero-free there, with a pole at 1. The numerator $L(s,\rho)$ is entire and of polynomial growth in fixed vertical strips by Hecke theory. Zeros of the cubic function therefore do not obstruct continuation of H in this half-plane.

For clarity about bounds, the reciprocal of either Dirichlet factor has polynomial growth on $\sigma \geq 1/2+\delta$. At large heights take a zero-free disk centered at $2+it$ with left edge $1/2+\delta/2$. On it the real part of the logarithm is $O_\delta(\log(2+|t|))$ by the polynomial bound for the function. The branch chosen by the Euler product is bounded at the center. Borel–Carathéodory on the smaller disk with left edge $1/2+\delta$ gives a polynomial bound for the reciprocal. Compact heights follow by continuity. The argument applies also to a fixed zero-free Hecke factor whenever that hypothesis is specified. Removed Euler factors cause no singularity in this region.

For $W \in C_c^\infty((1,2))$ Mellin inversion and contour shift now give

$$\sum_n f_*(n) W(n/X) = O_{W,\epsilon}(X^{1/2+\epsilon}).$$

The same proof works for $W(t) = e^{-t}$ since its Mellin transform is $\Gamma(s)$. Test how much small-factor removal this can tolerate. Restrict to integers with every prime factor $> z$, taking $n = 1$ as eligible. This multiplies $H(s)$ by $P_z(s) = \prod_{q\in\mathcal{B},\ 7\neq q\leq z}(1-3q^{-s})^{-1}$. For fixed $1/2 < \sigma < 1$, on $\Re s \geq \sigma$ we have $\log|P_z(s)| \ll_\sigma z^{1-\sigma}$. Consequently for every fixed $A > 0$, $z = (\log X)^A$, the restricted sum is

$$O_{A,W,\epsilon}\big(X^{\max(1/2,\,1-1/A)+\epsilon}\big).$$

Indeed choose a contour just to the right of the maximum, making the multiplier $X^{o(1)}$. In particular the square-root estimate survives at the cutoff $(\log X)^2$.

At $z = \sqrt{2X}$ only primes can contribute on the support of W. For nonnegative W of nonzero integral, the sum then has main term $-X\int W(t)\,dt/\log X$, from the density 1/3 of $\mathcal{B}$. The small-factor removal has changed the size of the answer. On the real axis $\log P_z(\sigma) \gg_\sigma z^{1-\sigma}/\log z$, by using the bad primes in $[z/2, z]$. Thus the same absolute estimate of the finite Euler product loses more than any power of X at this cutoff. The signed bound for all integers gives no centered prime error here.

3. A test using nonnegative coefficients

Let $U(s) = \sum_n |f_*(n)| n^{-s}$. On $\Re s > 1$,

$$H(s)U(s) = \prod_{\substack{q \in \mathcal{B} \\ q \neq 7}} (1 - 9q^{-2s}).$$

The product extends without zeros to $\Re s > 1/2$; it and its inverse are bounded on $\Re s \geq 1/2 + \delta$. Under Dirichlet GRH the zero of H at 1 is simple, since the nonprincipal Hecke L-function does not vanish at 1. Therefore U has a simple pole at 1 with positive residue C. In $\Re s > 1/2$ its other poles are exactly the zeros of $L(s, \rho)$, with their multiplicities. There is no cancellation from small Euler factors under the convention used here.

This gives a test of the strength of the unsigned estimate. Under Dirichlet GRH the following are equivalent:

$$\text{RH for } K = \mathbb{Q}(\sqrt[3]{2});$$
$$\sum_{n \geq 1} |f_*(n)| e^{-n/X} = CX + O_\epsilon(X^{1/2+\epsilon}) \quad \text{for every } \epsilon > 0.$$

In the forward direction $\zeta_K = \zeta L(\rho)$ makes the cubic factor zero-free on the desired half-plane. Shift the Mellin contour for $\Gamma(s)U(s)$, picking up its pole at 1. The preceding reciprocal bound supplies sufficient growth control. Conversely, Mellin transform the displayed estimate in X using $X^{-s-1}\,dX$. The integral below $X = 1$ is entire by exponential decay at zero. Subtracting CX on $X \geq 1$ continues $\Gamma(s)U(s) - C/(s-1)$ holomorphically to $\Re s > 1/2$. Since Γ has no zeros there, $L(s, \rho)$ cannot have a zero there. Its functional equation and Dirichlet RH give the claim for K.

The denominator comparison is exact and the smoothed signed bound uses only the Dirichlet hypothesis. The positive square-root estimate, which would supply information on support without sign cancellation, already requires the missing cubic zero-free result. The preceding argument proves the equivalence, not either side. No improvement to the known cubic zero-free region has been obtained. The product manipulations use standard Frobenian Euler-product methods; priority for the specialization has not been established.

Sources. Earlier conductor-108 and elliptic-denominator notes; fixed Artin representation and character values; conditional bounds for logarithms of L-functions; analytic continuation of Frobenian Euler products.