Dirichlet descent in the conductor-108 cubic problem

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Let $K = \mathbb{Q}(\sqrt[3]{2})$, $F = \mathbb{Q}(\sqrt{-3})$, and let τ be complex conjugation. Away from ramification write $\kappa_a(P) = (a/P)_6$ for $a \in F^\times$. The target is $\eta = \kappa_4$: its value is $(2/P)_3$, the Kummer character of $F(\sqrt[3]{2})/F$. Either choice of the cubic character gives the same L-function by conjugating ideals. Induction to $\mathbb{Q}$ gives the standard S_3 representation and $\zeta_K(s) = \zeta(s)L_F(s,\eta)$. See the conductor-108 entry.

1. Every denominator fails the norm test

In OpenAI's September preprint, §7.1 removes frequencies meeting a fixed finite set S containing the primes over 2 and 3. Each surviving frequency is ua^6, with u sixth-power-free, coprime to S, and carrying its unit. The scalar quotient in (7.13), which precedes the correction in (16.3), is

$$\zeta_F^S(6z)\frac{L_F^S(w,\kappa_u)}{L_F^S(x,\eta\,\overline{\kappa_u})}.$$

Here the overbar in the denominator conjugates the character value. Primitive characters are used in the descent test. Finite Euler deletions keep their factors nonzero at positive real part.

Descent criterion. κ_a is the pullback of a rational Dirichlet character under the ideal norm precisely when $\mathrm{N}a = a\bar{a}$ is a sixth power in $\mathbb{Q}_{>0}$. Indeed, $\kappa_a^\tau = \kappa_{\bar{a}}^{-1}$ by conjugating the residue symbol. Thus invariance is equivalent to $\kappa_{a\bar{a}} = 1$, or $a\bar{a} \in F^{\times 6}$ by Kummer theory and Frobenius density (see also Lemma 4.1 of the preprint). A sixth root in F of a positive rational number lies on a ray of argument $j\pi/3$. Multiplying by a unit makes it real, hence rational. To see sufficiency of invariance, extend the invariant finite Galois character of G_F to $G_\mathbb{Q} = G_F \rtimes \langle\tau\rangle$ by giving τ value 1. This finite abelian character is Dirichlet by Kronecker–Weber, and restriction corresponds to norm pullback. The two Dirichlet factors of the resulting Hecke function are given in Proposition 11.3 of the preprint.

Apply this with $a = 4/u$ for the denominator. Then

$$\mathrm{N}(4/u) = 16/\mathrm{N}u, \qquad v_2(16/\mathrm{N}u) = 4$$

for every allowed u, including $u = 1$. None descends. In fact $(\kappa_a/\kappa_a^\tau)^2 = (\mathrm{N}a/\cdot)_3$ has the same nontrivial cubic inertia at 2 for every row. The odd rational number $\mathrm{N}u$ is a cube in $\mathbb{Q}_2$ by Hensel's lemma, and 16 differs from 2 by a cube. Adjoining a cube root of 2 over F_2 (the completion at its unique prime over 2) is totally ramified of degree three by Eisenstein.

Pairing a character ψ with ψ^τ gives $L_F(s,\psi^\tau) = L_F(s,\psi)$ by reindexing ideals. Although $\psi\psi^\tau$ descends, its L-function is a different operation from multiplying the two L-functions. More generally induction of each denominator character to $G_\mathbb{Q}$ is irreducible: its restriction has the two distinct constituents ψ and ψ^τ, interchanged by τ. Direct sums of these inductions cannot be sums of Dirichlet characters, by character uniqueness and Chebotarev. This tests factorization and leaves open analytic cancellation between rows.

2. Counting exactly the numerator rows covered by Dirichlet GRH

For the numerator the criterion is $\mathrm{N}u = t^6$. Let $S_{\mathbb{Q}}$ be the rational primes below S. At a split prime $q \mid t$ the exponents above q are in $[0,5]$ and have sum six, so they are $(1,5),(2,4),(3,3),(4,2),(5,1)$. At an inert prime $q \mid t$ the sole exponent is three. Hence t is squarefree, and both ideals over any split prime divisor of t occur. This requires $(t, \prod_{q\in S_{\mathbb{Q}}} q) = 1$, even if only one place over a split prime was removed. Conversely each listed factorization determines one ideal, with six element generators, since F has class number one. Put $\omega_1(t) = \#\{q \mid t : q \equiv 1 \pmod 3\}$. The exact number of eligible numerator indices of norm at most U is

$$6 \sum_{\substack{t \le U^{1/6} \\ q\mid t \Rightarrow q \notin S_{\mathbb{Q}}}} \mu^2(t) 5^{\omega_1(t)}.$$

Its asymptotic is $C_S U^{1/6} (\log U)^2/12$, with

$$C_S = \lim_{s\downarrow 1} (s-1)^3 \prod_{q \notin S_{\mathbb{Q}}} (1 + (3 + 2\chi_{-3}(q)) q^{-s}) > 0.$$

Indeed the product equals $\zeta(s)^3 L(s,\chi_{-3})^2 G_S(s)$, where all linear terms of the good Euler factors of G_S cancel. Its tail converges locally absolutely for $\Re s > 1/2$ and $G_S(1) > 0$. Finite factors at removed primes are holomorphic near $\Re s \ge 1$. The nonnegative series thus has a triple pole at one and continues across the rest of that line. The Ikehara–Delange theorem gives $\frac{1}{2} C_S X (\log X)^2$ for its coefficient sum up to X. Use $X = U^{1/6}$ and count the six units.

Finite calculation. For $S_{\mathbb{Q}} = \{2,3\}$ and $t = 1,5,7,11,13,35,91$, direct enumeration of $a^2 - ab + b^2 = t^6$ gave respectively $6,6,30,6,30,30,150$ rows after the sixth-power-free restriction. One can reproduce the count by looping over $3b^2 \le 4t^6$, requiring $4t^6 - 3b^2 = z^2$ and testing the integral values $a = (b \pm z)/2$ without duplication. At each split $q \mid t$, require $q \mid a, b$ to exclude the two forbidden endpoint valuations. At inert primes the norm already fixes the valuation.

Analytic consequence. Dirichlet GRH applies to $\zeta_F = \zeta L(\cdot, \chi_{-3})$ and to the listed primitive numerator characters. It puts no physical denominator into Proposition 11.3's norm-factorization case. Section 8.1's detector includes the denominator in its family for each row. A count of unweighted indices gives no bound for their analytic contribution. No improvement in the zero-free half-plane for the cubic factor follows by this substitution.

Literature benchmark. Baiying Liu's 20 November preprint states a zero-free half-plane $\Re s > \theta_G$ with $0.8733326374459049352 < \theta_G < 0.8733326374459051924$ for unitary Hecke characters over every number field. This includes our fixed character. We checked its stated scope. The zero-free theorem and its dependencies have not been independently verified here. Taking its theorem as input also gives a benchmark for the original congruence. Let $E = F(\sqrt[3]{2})$ and sum $\log q$ over primes $q > 3$ up to X where $z^3 = 2 \bmod q$ has three roots, obtaining $\Theta(X)$. Apply the fixed-field prime ideal explicit formula to ζ_E, covered by Liu's theorem with the trivial character of E. The zeros up to height $T \asymp X$ have reciprocal absolute values summing to $O_E(\log^2 X)$. In this Galois extension the degree-one terms give $6\Theta(X)$, with prime powers, higher degrees and ramification contributing $O_E(\sqrt{X})$. Thus $6\Theta(X) = X + O_{E,\epsilon}(X^{\theta_G + \epsilon})$. This is a consequence of the cited bound.

3. A sum where cubing the character does help

Let S_0 contain just the places above 2 and 3. For $X > 0$ define the real sum

$$A(X) = \sum_{\substack{\mathcal{I} \text{ integral cube-free} \\ (\mathcal{I},6)=1}} \eta(\mathcal{I}) e^{-\mathrm{N}\mathcal{I}/X}.$$

Reality follows by pairing conjugate ideals. Under Dirichlet GRH,

$$A(X) \ll_\epsilon X^{1/6+\epsilon}, \qquad A(X) = \Omega_+(X^{1/6}) \quad \text{and} \quad A(X) = \Omega_-(X^{1/6}).$$

These are estimates for an ideal sum. Here is a proof for this specific character and the fixed exponential weight.

Set $D(s) = L_F^{S_0}(s,\eta)/\zeta_F^{S_0}(3s)$. The local identity $1 + v + v^2 = (1-v^3)/(1-v)$ with $v = \eta(P)(\mathrm{N}P)^{-s}$ gives $D(s)$ as the Dirichlet series in $\Re s > 1$. This uses $\eta^3 = 1$. Thus $M(s) = \Gamma(s)D(s)$ is the Mellin integral of $A(X)$ against $X^{-s-1}dX$. Under Dirichlet GRH its denominator is zero-free for $\Re s > 1/6$, with an allowed pole at $1/3$ of the denominator giving a zero of the reciprocal. In any buffered half-strip the reciprocal has polynomial growth: at large heights use the Euler product at $2+it$, a polynomial upper bound and Borel–Carathéodory for the logarithm on zero-free disks. The fixed numerator has polynomial growth by its functional equation, and the finite deletions and their inverses are bounded on buffered positive lines. Exponential decay of Γ permits the contour shift to $1/6 + \epsilon$.

Why some poles necessarily survive. For either fixed degree-two function $L_F(s,\eta)$ or $\zeta_F(s)$ the positive-height zero count is $(T/\pi)\log T + O(T)$ with multiplicity, by the degree-two Riemann–von Mangoldt formula. The denominator therefore has $(3T/\pi)\log T + O(T)$ zeros on $\Re s = 1/6$ up to height T under Dirichlet GRH. Each numerator zero there has a partner at $5/6$ and the same height and multiplicity by the functional equation. Hence the uncancelled pole multiplicity of M on the line up to T is at least $(5/(2\pi))T\log T - O(T)$. No pole occurs at the real point $\alpha = 1/6$: $\zeta(1/2) \neq 0$ by the alternating series, and $L(1/2, \chi_{-3}) > 0$ by grouping terms modulo three.

Choose a pole $\alpha + i\tau$, $\tau > 0$, and let $r \neq 0$ be its leading Laurent coefficient. If $A(X) \leq cX^\alpha$ eventually, the Mellin integral of the nonnegative tail $cX^\alpha - A(X)$ at $\sigma + i\tau$ has absolute value at most its value at real $\sigma > \alpha$. The omitted compact range and the integral of $A(X)$ near zero are holomorphic there. Multiply by $\sigma - \alpha$ and let $\sigma \downarrow \alpha$. The real side tends to c. The left tends to $|r|$ for a simple pole and to infinity for a multiple pole. Thus no such bound holds with $0 < c < |r|$. Applying the same argument to $-A$ gives the negative excursions.

Liu's Theorem 4.1 and Propositions 4.2–4.4 already study the general quotient for centered cube-free sums. Here the pole count and oscillation concern one fixed field, character and weight. The target cubic L-function occurs in the numerator. Its zeros are compatible with the estimate. These hand deductions carry no claim of priority or Lean verification.