A finite range for a prime race attached to cube roots of two

Manuscript revised for review, 25 November 2026.

A working note for IIS-Q-0001. Lennart Lopin's cubic-field project. Calculation by Eisenquill (OpenAI Codex).

Let a_p be the number of roots of $z^3 = 2 \bmod p$ minus one, for $p > 3$. Thus $a_p = 2, 0, -1$ for three, one, or zero roots. Define

$$S(X) = \sum_{p>3} a_p (\log p) e^{-p/X}.$$

Write $D(X) = -S(X)$. With weight $(\log p)e^{-p/X}$, this is the total for primes with no root minus twice the total for primes with three roots.

Theorem 1 (unconditional). For every real X in the stated interval,

$$\boxed{D(X) \geq 0.09\sqrt{X} \qquad (10^{50} \leq X \leq 10^{110}).} \tag{1}$$

The field-specific computational input is Tollis's published verification of the low zeros. The proof combines it with explicit estimates for the two reduced residue classes modulo three. The accompanying Python checker verifies the rounded constants in that deduction.

Corollary 2 (additional zero-free input). If all nontrivial zeros of the conductor-108 cubic factor have real part at most $7/8$, the same conclusion holds for $10^{50} \leq X \leq 10^{147}$.

The input specific to this field. Write $K = \mathbb{Q}(\sqrt[3]{2})$ and $F(s) = \zeta_K(s)/\zeta(s)$. This is the entire standard S_3 Artin L-function, induced from a cubic character of $\mathbb{Q}(\sqrt{-3})$. The discriminant of K is -108. The completed quotient $\mathcal{L}(s) = Q^s \Gamma(s) F(s)$, where $Q = \sqrt{108}/(2\pi)$, is entire of order one and satisfies $\mathcal{L}(s) = \mathcal{L}(1-s)$. These facts also follow by taking the completed zeta quotient of the signature $(1,1)$ field and applying gamma duplication. Both ramified Euler factors of F are one, since $x^3 - 2$ at 2 and $(x-1)^3 - 2$ at 3 are Eisenstein. See the Artin record and corresponding newform.

Tollis (1997), Table 6.2, lists the exact polynomial $x^3 - 2$. For ζ_K the GRH height is reported as 94.26, the first positive ordinate as 2.82161, and the positive zero count through height 92 as 143. The last count is for the full Dedekind zeta function. Since $\zeta_K = \zeta F$ and ζ has no pole in the open strip, the nonreal zeros $\rho = \beta + i\gamma$ of $\mathcal{L}$ inherit $|\gamma| > 2.82$ and critical-line verification for $|\gamma| \leq 94$. Tollis checks completeness by a Turing argument. Below we exclude real zeros of the cubic factor separately.

A positive budget for all zeros. At real $s = 1, 2$, Hadamard factorization gives $\mathcal{L}'(s)/\mathcal{L}(s) = \sum_\rho \operatorname{Re}(1/(s-\rho))$. Indeed, in the centered variable $z = s - 1/2$, first factor out z^{m_0}, where m_0 is the possible central multiplicity. The remaining even Hadamard product pairs opposite zeros. Its logarithmic derivative together with $m_0/(s-1/2)$ accounts for every zero with multiplicity. Pairing conjugates gives the absolutely convergent real sum. Put $w_\rho = \operatorname{Re}(1/(2-\rho))$. Then $w_\rho \geq 1/(4+\gamma^2)$ and

$$\sum_\rho w_\rho = \log Q + 1 - \gamma_E + F'(2)/F(2) < 1.62. \tag{2}$$

Indeed $|F'(2)/F(2)| \leq 2\sum_{p \geq 5} \log p/(p^2-1)$, because every good prime-power trace has modulus at most two. The sum without the factor two is at most $\frac{25}{24}(\log(5)/25 + (1+\log(5))/10) < 0.34$, by summing over odd integers and using a decreasing-function integral bound.

Excluding a central zero. Write $M = \sum_\rho w_\rho$. In the Euler bound for (2), sum $\log p/(p^2-1)$ over $5 \leq p \leq K_0 = 10{,}000$ explicitly. The tail is at most $(1 + \log K_0)/(K_0(1 - K_0^{-2}))$, by integrating $(\log t)/t^2$ from K_0 to infinity. The bundled height-92 checker gives $M < 1.331 < 4/3$. Evenness forces any zero at $s = 1/2$ to have multiplicity at least two. Its entries alone would contribute $2/(2-1/2) = 4/3$, and all the weights are positive. Hence $\mathcal{L}(1/2) \neq 0$. Any other real zero in the strip would have a distinct reflected partner, whose combined

weight with it is $1/(2-\beta)+1/(1+\beta)\ge 4/3$. Thus the absolute-ordinate lower bound applies to every zero in the estimates below.

For a zero on the line,

$$\frac{|\Gamma(\rho)|}{w_\rho}\le \frac{9/4+\gamma^2}{3/2}\sqrt{2\pi}\,e^{-\pi|\gamma|/2}<0.208.$$

The function on the right decreases above 2.8. This follows from the exact identity $|\Gamma(1/2+it)|^2=\pi/\cosh(\pi t)$. Thus the total absolute contribution of all critical-line zeros to the exponentially weighted explicit formula is less than $0.34\sqrt{X}$.

If a zero is off the line, put $t=|\gamma|\ge 94$. Rotation of Euler's gamma integral for $1+\rho$ through angle $1.55\operatorname{sgn}(\gamma)$ gives $|\Gamma(\rho)|\le e^{-1.55t}/(t\cos^2(1.55))$, since $\Gamma(1+\beta)\le 1$. Using $\cos(1.55)>0.02$, we get

$$\frac{|\Gamma(\rho)|}{w_\rho}\le 2500(t+4/t)e^{-1.55t}\le 2500(94+4/94)e^{-1.55\cdot 94}<1.25\cdot 10^{-58}.$$

Consequently, if $\beta\le B$ for all zeros, the total absolute contribution of the off-line zeros is at most $3\cdot 10^{-58}X^B$ for $X\ge 1$. The unconditional critical-strip bound permits $B=1$.

Explicit formula with a bounded remainder. Define $b(p^k)=\operatorname{tr}(g_p^k)$ in the standard representation for $p>3$, and zero for $p=2,3$. Write $A(X)=\sum_{p,k\ge 1}b(p^k)\log p\,e^{-p^k/X}$, with p prime. Mellin inversion of $-\Gamma(s)(F'/F)(s)X^s$ and a shift from 2 to $-1/2$ give

$$A(X)=-\sum_\rho \Gamma(\rho)X^\rho-\log X+\log Q+\mathcal{L}'(1)/\mathcal{L}(1)+R(X),\qquad |R(X)|\le 100X^{-1/2}. \tag{3}$$

Here $s=0$ contributes a double pole because F has a simple trivial zero and Γ a simple pole. In the expansion the two Euler constants cancel. The positive real sum at $s=1$ is at most $2\sum w_\rho$, by $|\gamma|>2.8$ and direct comparison of the denominators. For details on the new contour, let $s=-1/2+it$ and $\psi=\Gamma'/\Gamma$. The digamma series gives $|\psi(z)|\le 5+4|t|$ for $1/2\le \operatorname{Re} z\le 3/2$, $|\Im z|=|t|$. Indeed, $\psi(z)=-\gamma_E+\sum_{n\ge 0}(z-1)/((n+1)(n+z))$. Use $\psi(s)=\psi(s+1)-1/s$, the functional equation, and $|F'(1-s)/F(1-s)|\le 2\sum_{n\ge 2}(\log n)n^{-3/2}<12$. They give $|F'(s)/F(s)|\le 40+8|t|$. Since $|\Gamma(s)|\le 2\sqrt{2\pi}e^{-\pi|t|/2}$, integration gives the stated 100. Horizontal integrals vanish along heights separated from zero ordinates by $\gg 1/\log(2+|t|)$, using the standard fixed-conductor local zero count and logarithmic derivative bound there. The zero sum is absolutely convergent by (2) and the gamma estimates.

Removing the prime powers without assuming cubic RH. For a prime $p\equiv 2 \pmod 3$, $p>3$, Frobenius is a transposition, so $b(p^2)=2$. For $p\equiv 1 \pmod 3$, $b(p^2)=2$ at the identity and $b(p^2)=-1$ at a three-cycle. The lower bound below makes the *aggregate* square contribution to D positive in our range. The correction for 2 removes the ramified prime from the class 2 mod 3, and neither reduced class contains 3. It follows that $\sum_{p\le v}b(p^2)\log p$ is bounded below by

$$2\theta(v;3,2)-\theta(v;3,1)-2(\log 2)\mathbf{1}_{v\ge 2}\ \ge\ 0.498v-3\cdot 10^{11}.$$

For $v\ge 8\cdot 10^9$ use Bennett–Martin–O'Bryant–Rechnitzer, Theorem 1.2, with error at most $v/(160\log v)$ in each progression. Below that endpoint even the bound $\theta(v;3,1)\le v\log(\max(2,v))$ suffices. Integrating against $(2v/X)e^{-v^2/X}$ gives a square contribution at least $0.44\sqrt{X}-3\cdot 10^{11}$. For the powers $k\ge 3$ use $e^{-u}\le u^{-2/5}$ to bound their total absolute contribution by

$$X^{2/5}\frac{2}{1-2^{-2/5}}\sum_{n\ge 2}\frac{\log n}{n^{6/5}}\le 250X^{2/5}.$$

For the last bound retain $n=2,3$ and majorize the rest by an integral from 3.

Proof of Theorem 1. Combining the inequalities gives, for $X \geq 1$,

$$S(X) \leq -0.10\sqrt{X} + 3 \cdot 10^{-58} X^B + 250X^{2/5} + 3 \cdot 10^{11} + \log X + 5 + 100X^{-1/2}.$$

After division by $\sqrt{X}$, the adverse terms other than the off-line zeros are $250X^{-1/10}$, $3 \cdot 10^{11} X^{-1/2}$, $(\log X + 5)X^{-1/2}$, and $100/X$, in addition to the constant allowance for critical-line zeros already included in 0.10. Each of the four displayed losses decreases for $X \geq 10^{50}$ and takes its largest value at 10^{50}. The off-line term $3 \cdot 10^{-58} X^{B-1/2}$ increases for $B > 1/2$. Taking $B = 1$ and $X = 10^{110}$ in this term bounds it by 0.003. The total loss from 0.10 is less than 0.01. This proves (1) throughout the real interval.

Proof of Corollary 2. With the additional bound $B = 7/8$, the off-line loss at 10^{147} is at most 0.00401. The same lower-endpoint estimates give the claimed margin. The September 2026 zero-free theorem states the additional input for finite-order Hecke characters over the Eisenstein field, including the character inducing F. We use its statement as an external result.

Verification notes. Tollis supplied the verified zero region. Our two scalar calculations use his published data. The companion audit takes 92 for the verification height and 2.82 as a lower bound on the first ordinate. Its interval calculation gives $D(X)/\sqrt{X} > 0.16362$ in Theorem 1 and > 0.16203 in Corollary 2. The independent reviewer reports verifying all 13 manifest entries and running both unmodified checkers in the 25 November reviewer ZIP. That replay reproduced the two margins above. The current height-92 checker also tests the central-zero budget and asserts each quoted margin directly.

Eventual-sign criterion. On RH for F, the off-line sum is empty, and (1) holds for every $X \geq 10^{50}$. Conversely, even $S(X) \leq 0$ for all sufficiently large X implies RH for F. Indeed, for $\operatorname{Re} s > 1$ the Mellin transform $\int_0^\infty S(X) X^{-s-1} dX$ is $\Gamma(s) \sum_{p>3} a_p \log p \, p^{-s}$. In $\operatorname{Re} s > 1/2$ this continues as $-\Gamma(s)F'(s)/F(s)$ plus a holomorphic function, since the higher prime-power series converges there. There are no real poles in that half-plane, by the zero data and Euler product. If $-S$ were eventually nonnegative, Landau's theorem for its Mellin transform would force any abscissa of convergence greater than $1/2$ to be a real singularity. The bounded initial part has entire transform because of exponential decay at zero. Hence there can be no zero of F to the right of $1/2$, and symmetry gives the assertion. With RH for ζ, this is exactly RH for K. A hypothetical zero to the right of the critical line above the verified heights contributes a factor $X^{\beta-1/2}$ after normalization. The finite-range estimate leaves the eventual sign open.

Relation to earlier work. The classical modulo-four criterion goes back to Hardy–Littlewood and Landau. Suzuki records the exponential weights, including $(\log p)e^{-p/X}$, in equations (16)–(17) [1]. Platt and Trudgian [2] developed Fujii's argument for the modulo-four race using gamma-weighted zero contributions. They verified the first 1,000 zeros and bounded the infinite tail to extend the range of smoothing exponents in an RH equivalence. Ng [3, §5.3.1] examined the normalized identity and three-cycle race for other S_3 fields with a sharp cutoff. His examples there have cubic discriminants $-23, -31, -59, -83$. The potential contribution here is the explicit finite range and positive margin for the field of discriminant -108. Priority for that specialization remains provisional.

1. M. Suzuki, *On variants of Chebyshev's conjecture*, Ramanujan J. 68, article 95 (2025). Corrected article, published correction.

2. D. Platt and T. Trudgian, *Fujii's development on Chebyshev's conjecture*, Int. J. Number Theory 15(3), 639–644 (2019), §§2–3. Journal, preprint.

3. N. Ng, *Limiting distributions and zeros of Artin L-functions*, Ph.D. thesis, University of British Columbia (2000), §5.3.1. Thesis.