Primes with three cube roots of two in a short interval

Working note for IIS-Q-0001. Lennart Lopin's cubic-field project, calculation by Eisenquill (OpenAI Codex).

Write $E = \mathbb{Q}(\sqrt[3]{2}, \sqrt{-3})$ and let $\Theta(x)$ sum $\log p$ over $p \le x$, $p > 3$, for which $z^3 \equiv 2 \pmod p$ has three solutions. The weight-one newform $f(z) = \eta(6z)\eta(18z)$ (here η is the Dedekind eta function) has level 108. Its good-prime coefficient a_p is the number of roots minus one, by the permutation character of S_3. Its Artin representation is the standard two-dimensional representation. The regular representation gives

$$\zeta_E(s) = \zeta(s)L(s, \chi_{-3})L(s, f)^2.$$

See the newform and the Artin representation.

Short-interval bound. Assume Dirichlet GRH. Fix $3/4 \le B < 1$ such that $L(s, f)$ has no zeros for $\Re s > B$. Uniformly for $x^{3/4} \le h \le x$ and each $\epsilon > 0$,

$$\boxed{6\big(\Theta(x+h) - \Theta(x)\big) = h + O_{B,\epsilon}\big(x^{B+\epsilon}(h/x)^{4B-3}\big)\,.} \tag{1}$$

The constants are for this fixed field. The published zero-density input allows this weight and level. The bound is a deduction from existing estimates and priority for the specialization has not been checked.

Which density estimate applies. Let $N_f(\sigma, V)$ count zeros with real part at least σ and $|\Im s| \le V$, with multiplicity. In de Faveri, Proposition A.1, p. 3013, take $Q = 2$ and retain the primitive character of conductor one, which twists f into itself. For $3/4 \le \sigma \le 1$ the auxiliary exponent $\min(3/(2-\sigma), 3/(3\sigma-1))$ is at most 3. Hence for each $\nu > 0$,

$$N_f(\sigma, V) \ll_\nu V^{4(1-\sigma)+\nu} \qquad (V \ge 2). \tag{2}$$

All logarithms can be absorbed in V^ν by taking the source's epsilon small. Even counting first without multiplicities would only cost $O(\log(2V))$ by the fixed-conductor local zero count. Under Dirichlet GRH the two rational factors have all their nontrivial zeros on the line $\Re s = 1/2$. Thus (2), with an additional factor two, bounds $N_E(\sigma, V)$ in the range used here.

Proof of (1). Put $Y = x/h \in [1, x^{1/4}]$. For the Dedekind prime-power sum ψ_E, the truncated explicit formula gives the main term h in the increment and the zero terms $-((x+h)^\rho - x^\rho)/\rho$ up to a good height $T \asymp x$. One may take each endpoint at the half-integer above its integer part and adjust h by $O(1)$. The fixed-field truncation costs $O(\log^C x)$ there: the prime-power coefficients are bounded by $6\Lambda(n)$, so truncated Perron at height $\asymp x$ costs $O(\log^2 x)$ at a half-integer. Use one height in $[x, x+1]$ that stays $\gg 1/\log x$ from zero heights, by the local zero count. The usual contour shift gives logarithmic horizontal and trivial-zero errors for both endpoints. See also Balog–Ono, Lemma 3.1 and equation (3.2). Each zero satisfies

$$\left|\frac{(x+h)^\rho - x^\rho}{\rho}\right| \ll h x^{\beta-1} \min\left(1, \frac{Y}{1+|\Im\rho|}\right), \qquad \beta = \Re\rho. \tag{3}$$

For $\beta \le 3/4$ the sum is $O(x^{3/4}\log^2 x)$ by the fixed-field zero count and (3). This fits (1) since $x^B Y^{-(4B-3)} \ge x^{3/4}$. For the other zeros, group real parts in strips of width $1/\log x$ with left endpoint $\sigma \ge 3/4$, and group $1+|\Im\rho|$ in dyadic ranges $\asymp V$. By (2) and (3), each group costs at most

$$O_\nu\Big(hx^{\sigma-1+\nu}V^{4(1-\sigma)}\min(1,Y/V)\Big).$$

For $a = 4(1-\sigma) \in [0,1]$ one has $V^a \min(1,Y/V) \le Y^a$ on both sides of $V = Y$. As $Y^4/x \le 1$ and $\sigma \le B$, each group costs $O_\nu(hx^\nu(Y^4/x)^{1-B})$. There are $O(\log^2 x)$ groups. Choose ν smaller than ϵ. The only unramified degree-one prime ideals of E lie in sets of six over completely split rational primes. Higher degrees and powers up to $2x$ together weigh $O(\sqrt{x}\log^2 x)$, by summing over rational prime powers of exponent at least two. The two ramified rational primes contribute $O(\log x)$. Removing these terms proves (1).

Benchmarks. At $h = x^{4/5}$ the error exponent in (1) is $3/5 + B/5 + \epsilon$. The seven-eighths Hecke theorem, applied to the cubic character over $\mathbb{Q}(\sqrt{-3})$, gives $31/40 + \epsilon$. Using $B = 0.87333264$ from Liu's 20 November preprint gives $0.774666528 + \epsilon$. We used the statement of Liu's theorem without independently auditing its proof. Both estimates hold for each sufficiently large starting point x. Compare the established short-interval Chebotarev theorem of Thorner, Theorem 1.1: with the identity class and abelian subgroup A_3, the fixed field has degree two and that theorem gives a logarithmic relative saving for $h \ge x^{4/5+\delta}$, $\delta > 0$.

Why this leaves the main question open. For $h = x$ formula (1) recovers the input exponent B. On a dyadic block, adding the absolute bounds for $\asymp x/h$ adjacent intervals gives $O(x^{B+\epsilon}(x/h)^{4(1-B)})$. The effect of one fixed zero ρ of real part B on one interval of length $h = o(x)$ is asymptotic in modulus to hx^{B-1}. This fits the bound in (1), even though its accumulated contribution can have size of order x^B. Thus the zero-density estimate gives a power saving for short-interval prime counts while leaving the global exponent and RH for $\mathbb{Q}(\sqrt[3]{2})$ unchanged. A collaborator can improve the global conclusion with an input that excludes or cancels such a contribution in the actual cubic prime count.