A finite range for a prime race attached to cube roots of two

A working note for IIS-Q-0001. Lennart Lopin's cubic-field project. Calculation by Eisenquill (OpenAI Codex).

Let a_p be the number of roots of $z^3 = 2 \bmod p$ minus one, for $p > 3$. Thus $a_p = 2, 0, -1$ for three, one, or zero roots. Define

$$S(X) = \sum_{p>3} a_p (\log p) e^{-p/X}.$$

Using existing published zero data and explicit bounds for the two prime classes modulo 3, we obtain, for every real X in the indicated interval,

$$\boxed{S(X) \le -0.09\sqrt{X} \qquad (10^{50} \le X \le 10^{110}).} \tag{1}$$

No GRH is required. If one uses the stated zero-free half-plane $\operatorname{Re} s > 7/8$ for finite-order Hecke characters over $\mathbb{Q}(\sqrt{-3})$, the upper endpoint in (1) becomes 10^{147}. The sum includes all primes, with exponential weights. The constants and endpoints are deliberately rounded. This is a hand deduction from the cited inputs. Priority for this specialization has not been established.

The input specific to this field. Write $K = \mathbb{Q}(\sqrt[3]{2})$ and $F(s) = \zeta_K(s)/\zeta(s)$. This is the entire standard S_3 Artin L-function, induced from a cubic character of $\mathbb{Q}(\sqrt{-3})$. The discriminant of K is -108. The completed quotient $\mathcal{L}(s) = Q^s \Gamma(s) F(s)$, where $Q = \sqrt{108}/(2\pi)$, is entire of order one and satisfies $\mathcal{L}(s) = \mathcal{L}(1-s)$. These facts also follow by taking the completed zeta quotient of the signature $(1,1)$ field and applying gamma duplication. Both ramified Euler factors of F are one, since $x^3 - 2$ at 2 and $(x-1)^3 - 2$ at 3 are Eisenstein. See the Artin record and corresponding newform.

Tollis (1997), Table 6.2, lists the exact polynomial $x^3 - 2$. For ζ_K the GRH height is reported as 94.26, the first positive ordinate as 2.82161, and the positive zero count through height 92 as 143. The last count is for the full Dedekind zeta function. We use only the following weakened data for the zeros $\rho = \beta + i\gamma$ of $\mathcal{L}$, with multiplicities: $|\gamma| > 2.8$ and every zero with $|\gamma| \le 94$ lies on the critical line. Indeed $\zeta_K = \zeta F$ and ζ has no pole in the open strip. Tollis checks completeness by a Turing argument. We use his published computation without rerunning it.

A positive budget for all zeros. At real $s = 1, 2$, Hadamard factorization gives $\mathcal{L}'(s)/\mathcal{L}(s) = \sum_\rho \operatorname{Re}(1/(s-\rho))$. For instance, center the product at $1/2$, group opposite zeros using evenness, and pair conjugates. The resulting real sums converge absolutely. Put $w_\rho = \operatorname{Re}(1/(2-\rho))$. Then $w_\rho \ge 1/(4+\gamma^2)$ and

$$\sum_\rho w_\rho = \log Q + 1 - \gamma_E + F'(2)/F(2) < 1.62. \tag{2}$$

Indeed $|F'(2)/F(2)| \le 2\sum_{p \ge 5} \log p/(p^2-1)$, because every good prime-power trace has modulus at most two. The sum without the factor two is at most $\frac{25}{24}(\log(5)/25 + (1+\log(5))/10) < 0.34$, by summing over odd integers and using a decreasing-function integral bound.

For a zero on the line,

$$\frac{|\Gamma(\rho)|}{w_\rho} \le \frac{9/4+\gamma^2}{3/2}\sqrt{2\pi}\, e^{-\pi|\gamma|/2} < 0.208.$$

The function on the right decreases above 2.8. This follows from the exact identity $|\Gamma(1/2+it)|^2 = \pi/\cosh(\pi t)$. Thus the total absolute contribution of all critical-line zeros to the exponentially weighted explicit formula is less than $0.34\sqrt{X}$.

If a zero is off the line, put $t = |\gamma| \ge 94$. Rotation of Euler's gamma integral for $1+\rho$ through angle $1.55\operatorname{sgn}(\gamma)$ gives $|\Gamma(\rho)| \le e^{-1.55t}/(t\cos^2(1.55))$, since $\Gamma(1+\beta) \le 1$. Using $\cos(1.55) > 0.02$, we get

$$\frac{|\Gamma(\rho)|}{w_\rho} \le 2500(t + 4/t)e^{-1.55t} \le 2500(94 + 4/94)e^{-1.55\cdot 94} < 1.25 \cdot 10^{-58}.$$

Consequently, if $\beta \le B$ for all zeros, the total absolute contribution of the off-line zeros is at most $3 \cdot 10^{-58} X^B$ for $X \ge 1$. One may always use $B = 1$. The September 2026 Hecke zero-free theorem supplies the optional $B = 7/8$ input for this cubic character. We use the stated theorem without independently checking its proof.

Explicit formula with a bounded remainder. Define $b(p^k) = \operatorname{tr}(g_p^k)$ in the standard representation for $p > 3$, and zero for $p = 2, 3$. Write $A(X) = \sum_{p,k\geq 1} b(p^k) \log p \, e^{-p^k/X}$, with p prime. Mellin inversion of $-\Gamma(s)(F'/F)(s)X^s$ and a shift from 2 to $-1/2$ give

$$A(X) = -\sum_{\rho} \Gamma(\rho) X^{\rho} - \log X + \log Q + \mathcal{L}'(1)/\mathcal{L}(1) + R(X), \qquad |R(X)| \leq 100 X^{-1/2}. \tag{3}$$

Here $s = 0$ contributes a double pole because F has a simple trivial zero and Γ a simple pole. In the expansion the two Euler constants cancel. The positive real sum at $s = 1$ is at most $2\sum w_\rho$, by $|\gamma| > 2.8$ and direct comparison of the denominators. For details on the new contour, let $s = -1/2 + it$ and $\psi = \Gamma'/\Gamma$. The digamma series gives $|\psi(z)| \leq 5 + 4|t|$ for $1/2 \leq \operatorname{Re} z \leq 3/2$, $|\Im z| = |t|$. Indeed, $\psi(z) = -\gamma_E + \sum_{n\geq 0}(z-1)/((n+1)(n+z))$. Use $\psi(s) = \psi(s+1) - 1/s$, the functional equation, and $|F'(1-s)/F(1-s)| \leq 2\sum_{n\geq 2}(\log n) n^{-3/2} < 12$. They give $|F'(s)/F(s)| \leq 40 + 8|t|$. Since $|\Gamma(s)| \leq 2\sqrt{2\pi} e^{-\pi|t|/2}$, integration gives the stated 100. Horizontal integrals vanish along heights separated from zero ordinates by $\gg 1/\log(2 + |t|)$, using the standard fixed-conductor local zero count and logarithmic derivative bound there. The zero sum is absolutely convergent by (2) and the gamma estimates.

Removing the prime powers without assuming cubic RH. For a prime $p \equiv 2 \pmod 3$, $p > 3$, Frobenius is a transposition, so $b(p^2) = 2$. For $p \equiv 1 \pmod 3$, $b(p^2)$ is 2 or -1. It follows that $\sum_{p\leq v} b(p^2) \log p$ is bounded below by

$$2\theta(v; 3, 2) - \theta(v; 3, 1) - 2(\log 2)\mathbf{1}_{v\geq 2} \;\geq\; 0.498v - 3 \cdot 10^{11}.$$

For $v \geq 8 \cdot 10^9$ use Bennett–Martin–O'Bryant–Rechnitzer, Theorem 1.2, with error at most $v/(160 \log v)$ in each progression. Below that endpoint even the bound $\theta(v; 3, 1) \leq v \log(\max(2, v))$ suffices. Integrating against $(2v/X)e^{-v^2/X}$ gives a square contribution at least $0.44\sqrt{X} - 3 \cdot 10^{11}$. For the powers $k \geq 3$ use $e^{-u} \leq u^{-2/5}$ to bound their total absolute contribution by

$$X^{2/5} \frac{2}{1 - 2^{-2/5}} \sum_{n\geq 2} \frac{\log n}{n^{6/5}} \leq 250 X^{2/5}.$$

For the last bound retain $n = 2, 3$ and majorize the rest by an integral from 3.

Range and scope. Combining the inequalities gives, for $X \geq 1$,

$$S(X) \leq -0.10\sqrt{X} + 3 \cdot 10^{-58} X^B + 250 X^{2/5} + 3 \cdot 10^{11} + \log X + 5 + 100 X^{-1/2}.$$

On dividing by $\sqrt{X}$, the power and constant remainders decrease for $X \geq 10^{50}$. The high-zero term is bounded by 0.003 through 10^{110} for $B = 1$, and by 0.00401 through 10^{147} for $B = 7/8$. This proves (1) and the stated extension.

On RH for F, the off-line sum is empty, and (1) holds for every $X \geq 10^{50}$. Conversely, even $S(X) \leq 0$ for all sufficiently large X implies RH for F. Indeed, for $\operatorname{Re} s > 1$ the Mellin transform $\int_0^\infty S(X) X^{-s-1} dX$ is $\Gamma(s) \sum_{p>3} a_p \log p \, p^{-s}$. In $\operatorname{Re} s > 1/2$ this continues as $-\Gamma(s)F'(s)/F(s)$ plus a holomorphic function, since the higher prime-power series converges there. There are no real poles in that half-plane, by the zero data and Euler product. If $-S$ were eventually nonnegative, Landau's theorem for its Mellin transform would force any abscissa of convergence greater than $1/2$ to be a real singularity. The bounded initial part has entire transform because of exponential decay at zero. Hence there can be no zero of F to the right of $1/2$, and symmetry gives the assertion. With RH for ζ, this is exactly RH for K. The exponential sign criterion has a classical predecessor for the modulo-four race, as discussed by Suzuki, formulas (16)–(17). An off-line zero with $\beta > 1/2$ still carries a factor $X^{\beta - 1/2}$ after normalization. The finite range gives no global improvement of the zero-free exponent or of the sharp splitting-prime count. No Lean formalization or independent peer review of this deduction is claimed.